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Generalized Gelfand invariants of quantum groups

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Abstract. Generalized Gelfand invariants of quantum groups are explicitly constructed, using a general procedure given in an earlier publication together with the Kirillov–Reshetikhin formula for universal R-matrices. As examples, invariants of $U_q(so(5))$ are considered.

1. Introduction

In an earlier publication [1], we presented a general method for constructing quantum group invariants using universal R-matrices, and determined their eigenvalues in arbitrary irreducible highest weight representations. These invariants reduce to (slight variations of) the Gelfand invariants of the corresponding simple Lie algebras in the $q \to 1$ limit. (For a discussion of Gelfand invariants see [2, 3] and especially [4].) As we demonstrated with the example of $U_q(sl(m))$, this method enables one to construct explicitly the generalized Gelfand invariants of a quantum group whenever the universal R-matrix is known.

Recently, Kirillov and Reshetikhin [5] have developed an explicit formula for universal R-matrices of quantum groups by studying the associated q-Weyl groups. The aim of this paper is to apply their formula to the general method developed in [1], and thus to obtain the generalized Gelfand invariants of quantum groups in explicit form.

As a concrete example, we study the simplest non-A_N-type quantum group, $U_q(so(5))$, in some detail. Previously, the generalized Gelfand invariants of the quantum groups $U_q(A_N)$ were studied quite extensively [1, 6–8], especially those of $U_q(A_1)$ and $U_q(A_2)$ [8]. However, none of the invariants of other quantum groups were given explicitly before. The case of $U_q(so(5))$ is of particular interest because it is the simplest quantum group arising from the deformation of the universal enveloping algebra of a Lie algebra whose roots are not all of equal length, i.e. one that is not simply laced. Consequently, this case exhibits features which the A_N-type quantum groups do not. For example, the quantum Weyl group of $U_q(so(5))$ differs markedly from that of $U_q(A_N)$.

2. Quantum groups

Recall that [6, 7, 9] corresponding to each simple Lie algebra g of rank r there is a quantum group $U_q(g)$ generated by simple elements e_i, f_i and Cartan elements h_i, for $i = 1, 2, \ldots, r$. Let $\{\alpha_i \mid i = 1, 2, \ldots, r\}$ be a set of simple roots of g, and let (a_{ij}) be the

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corresponding Cartan matrix, where $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$, with $(\cdot,\cdot)$ the invariant bilinear form on H^*, normalized in such a way that the maximum value of (α_i,α_i) for $i=1,2,\ldots,r$ is 2. Here H^* is the dual of the vector space H spanned by the h_i's. Then e_i, f_i, h_i are required to satisfy the following relations:

$$\begin{aligned}
&[h_i,h_j]=0\\
&[h_i,e_j]=(\alpha_i,\alpha_j)e_j \qquad [h_i,f_j]=-(\alpha_i,\alpha_j)f_j \qquad \forall i,j\\
&\sum_{t=0}^{1-a_{ij}}(-1)^t\begin{bmatrix}1-a_{ij}\\ t\end{bmatrix}_{q_i} e_i^{1-a_{ij}-t}e_je_i^t=0 \qquad i\neq j\\
&\sum_{t=0}^{1-a_{ij}}(-1)^t\begin{bmatrix}1-a_{ij}\\ t\end{bmatrix}_{q_i} f_i^{1-a_{ij}-t}f_jf_i^t=0 \qquad i\neq j
\end{aligned} \tag{1}$$

where $0\neq q\in\mathbb{C}$, $q_i=q^{(\alpha_i,\alpha_i)/2}$, and

$$\begin{bmatrix}m\\ n\end{bmatrix}_q=\begin{cases}[m]_q!/([m-n]_q![n]!) & m>n>0\\ 1 & n=0,m\end{cases}$$

with

$$[m]_q=(q^m-q^{-m})/(q-q^{-1})$$
$$[m]_q!=[m]_q[m-1]_q\ldots[1]_q.$$

Then $U_q(g)$ has the structure of a Hopf algebra with the co-multiplication $\Delta: U_q(g)\rightarrow U_q(g)\otimes U_q(g)$

$$\begin{aligned}
&\Delta(e_i)=e_i\otimes q^{h_i/2}+q^{-h_i/2}\otimes e_i\\
&\Delta(f_i)=f_i\otimes q^{h_i/2}+q^{-h_i/2}\otimes f_i\\
&\Delta(h_i)=h_i\otimes 1+1\otimes h_i \qquad \forall i
\end{aligned}$$

co-unit $\varepsilon: U_q(g)\rightarrow\mathbb{C}$

$$\varepsilon(1)=1 \qquad \varepsilon(h_i)=\varepsilon(e_i)=\varepsilon(f_i)=0 \qquad \forall i$$

and antipode $S: U_q(g)\rightarrow U_q(g)$

$$S(1)=1 \qquad S(h_i)=-h_i \qquad S(e_i)=-q_ie_i \qquad S(f_i)=-q_i^{-1}f_i.$$

For each $U_q(g)$, there exists an invertible element $R\in U_q(g)\otimes U_q(g)$, which is called the universal R-matrix of $U_q(g)$, such that

$$\begin{aligned}
&R\Delta(a)=\Delta'(a)R \qquad \forall a\in U_q(g)\\
&(\Delta\otimes id)R=R_{13}R_{23} \qquad (id\otimes\Delta)R=R_{13}R_{12}
\end{aligned}$$

where $\Delta'=T\cdot\Delta$ with $T: U_q(g)\otimes U_q(g)\rightarrow U_q(g)\otimes U_q(g)$ defined by

$$T(a\otimes b)=b\otimes a \qquad \forall a,b\in U_q(g)$$

and we say that $(U_q(g),R)$ constitutes a quasitriangular Hopf algebra [9].

An explicit formula for the universal R-matrix has been obtained by Kirillov and Reshetikhin [5] using quantum Weyl group techniques, which reads

$$\begin{aligned}
R=q^{\sum_{\mu=1}^{r}H_\mu\otimes H_\mu}\sum_{\{n\}}\prod_{p=1}^{K}\frac{(1-q^{-2}(p))^{n_p}}{(n_p)_{q(p)}!}&(E(1))^{n_1}(E(2))^{n_2}\ldots(E(K))^{n_K}\\
&\otimes(F(1))^{n_1}(F(2))^{n_2}\ldots(F(K))^{n_K}
\end{aligned} \tag{2}$$

where

$$n_p \in \mathbb{Z}^+ \qquad \forall p$$

$$K = \tfrac{1}{2}(\dim g - r)$$

$$(n)_q! = \prod_{i=1}^{n} \frac{1-q^{-2i}}{1-q^{-2}}$$

$$q(p) = q^{(\alpha(p),\alpha(p))/2}$$

with $\alpha(p) \in H^*$ defined by $[h_i, E(p)] = (\alpha_i, \alpha(p))E(p)$, $\forall i$. In (2), H_μ, $\mu = 1, 2, \ldots, r$ are the Cartan elements such that

$$\sum_{\mu=1}^{r} \alpha(H_\mu)\beta(H_\mu) = (\alpha, \beta) \qquad \forall \alpha, \beta \in H^*$$

where $\alpha(H_\mu)$, $\beta(H_\mu) \in \mathbb{C}$ are the evaluations of $\alpha, \beta \in H^*$ on $H_\mu \in H$. The $E(p)$, $F(p) \in U_q(g)$, $\forall p$ are defined explicitly in [5]. Due to the lack of space we do not describe them here and refer to [5] for details. However, it is worth pointing out that in the limit $q \to 1$, $E(p)$ and $F(p)$ reduce respectively to the raising and lowering operators of g shifting weight by $\alpha(p)$.

3. Invariants of quantum groups

Define

$$\Gamma = R^T R \qquad R^T = T(R).$$

It is important to observe that $\Gamma \neq 1 \otimes 1$ since $U_q(g)$ is not triangular, though quasitriangular. Let π_{Λ_0} be a non-trivial representation of $U_q(g)$ afforded by the finite-dimensional irreducible module $V(\Lambda_0)$ with highest weight Λ_0, and assume that $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the distinct weights of $V(\Lambda_0)$ with multiplicities $d_1, d_2, \ldots, d_N$ respectively. Now we construct [1, 10]

$$C^{\Lambda_0} = \mathrm{tr}_{\pi_{\Lambda_0}}[(id \otimes \pi_{\Lambda_0})(1 \otimes q^{2h_\rho})\Gamma] \in U_q(g) \tag{3}$$

where the trace is taken over the irreducible representation π_{Λ_0}, and $h_\rho \in H \subset U_q(g)$ is the Cartan element such that $\beta(h_\rho) = (\beta, \rho)$, $\forall \beta \in H^*$, with

$$\rho = \tfrac{1}{2} \sum_{p=1}^{K} \alpha(p).$$

Using results from [10] we proved in [1] that C^{Λ_0} belongs to the centre of $U_q(g)$, and when acting on the irreducible $U_q(g)$-module $V(\Lambda)$ with highest weight Λ, takes the eigenvalue

$$\chi_\Lambda(C^{\Lambda_0}) = \sum_{l=1}^{N} d_l q^{2(\Lambda+\rho,\lambda_l)}. \tag{4}$$

Observe that when $V(\Lambda)$ is trivial, i.e. $\Lambda = 0$, then $\chi_\Lambda(C^{\Lambda_0})$ coincides with the q-dimension of the reference representation π_{Λ_0}, i.e.

$$\chi_0(C^{\Lambda_0}) = D_q(\Lambda_0)$$

with

$$D_q(\Lambda_0)=\prod_{p=1}^{K}\frac{q^{(\Lambda_0+\rho,\alpha(p))}-q^{-(\Lambda_0+\rho,\alpha(p))}}{q^{(\rho,\alpha(p))}-q^{-(\rho,\alpha(p))}}.$$

Define a new central element $C_L^{\Lambda_0}\in U_q(g)$ by

$$C_L^{\Lambda_0}=[C^{\Lambda_0}-D_q(\Lambda_0)]/(q-q^{-1})^2.$$

In the limit $q\to 1$, $C_L^{\Lambda_0}$ reduces to $\frac{1}{2}l_{\Lambda_0}C_L$, where C_L is the quadratic Casimir of the Lie algebra g, and l_{Λ_0} is the Dynkin index [11] of the representation $\pi_{\Lambda_0}|_{q=1}$ of g, given by $l_{\Lambda_0}=(\Lambda_0+2\rho,\Lambda_0)\dim\pi_{\Lambda_0}/\dim g$. Using equation (2) we obtain

$$\begin{aligned}C_L^{\Lambda_0}=(q-q^{-1})^{-2}\,\mathrm{tr}_{\pi_{\Lambda_0}}\Bigg\{&q^{2\Sigma_{\mu=1}^{r}H_\mu\pi_{\Lambda_0}(H_\mu)}\\&\times\sum_{\{m\},\{n\}}\prod_{k,l=1}^{K}\left[\frac{(1-q^{-2}(k))^{n_k}(1-q^{-2}(l))^{m_l}q^{-n_kn_l(\alpha(k),\alpha(l))}}{(n_k)_{q(k)}!(m_l)_{q(l)}!}\right]\\&\times q^{-\Sigma_{p=1}^{K}n_ph_{\alpha(p)}}(F(1))^{n_1}\ldots(F(K))^{n_K}(E(1))^{m_1}\ldots(E(K))^{m_K}\\&\times\pi_{\Lambda_0}[q^{\Sigma_{p=1}^{K}(n_p+1)h_{\alpha(p)}}(E(1))^{n_1}\ldots(E(K))^{n_K}(F(1))^{m_1}\ldots(F(K))^{m_K}]\\&-\pi_{\Lambda_0}(q^{2h_\rho})\Bigg\}.\end{aligned}\tag{5}$$

We emphasize that equation (5) is an explicit formula. It is seemingly complex, but in fact very easy to handle, especially when we choose π_{Λ_0} to be a minimal representation of $U_q(g)$ (if $U_q(g)$ admits any), since now all the n_p, m_p's can only take values 0 and 1.

Now we consider the co-multiplication of $C_L^{\Lambda_0}$. Since

$$(\Delta\otimes id)\Gamma=(R^T)_{23}(R^T)_{13}R_{13}R_{23}$$

we immediately obtain

$$\begin{aligned}\Delta(C_L^{\Lambda_0})=(q-q^{-1})^{-2}\,\mathrm{tr}_{\pi_{\Lambda_0}}\{&(id\otimes id\otimes\pi_{\Lambda_0})(1\otimes 1\otimes q^{2h_\rho})\\&\times[(R^T)_{23}(R^T)_{13}R_{13}R_{23}-1\otimes 1\otimes 1]\}.\end{aligned}\tag{6}$$

By using (2) in (6), $\Delta(C_L^{\Lambda_0})$ can be computed explicitly. Now let

$$A=\Delta(C_L^{\Lambda_0})-C_L^{\Lambda_0}\otimes 1-1\otimes C_L^{\Lambda_0}\tag{7}$$

and define

$$I_m^{\Lambda_0}=\mathrm{tr}_{\pi_{\Lambda_0}}\{(id\otimes\pi_{\Lambda_0})[(1\otimes q^{2h_\rho})A^m]\}\qquad 1<m\in\mathbb{Z}^+.\tag{8}$$

Then $I_m^{\Lambda_0}$ belongs to the centre of $U_q(g)$; when acting on the irreducible $U_q(g)$ module $V(\Lambda)$ with highest weight Λ, it takes the eigenvalue

$$\chi_\Lambda(I_m^{\Lambda_0})=\sum_{l=1}^{N}d_l[\alpha_l(\Lambda)]^mD_q(\Lambda+\lambda_l)/D_q(\Lambda)\tag{9}$$

where

$$\alpha_l(\Lambda)=(q-q^{-1})^{-2}\sum_{j=1}^{N}d_j\{q^{2(\Lambda+\lambda_l+\rho,\lambda_j)}+q^{2(\rho,\lambda_j)}-q^{2(\Lambda+\rho,\lambda_j)}-q^{2(\Lambda_0+\rho,\lambda_j)}\}\tag{10}$$

as can be proved by the same method used in [1]. In the limit $q\to 1$, the $I_m^{\Lambda_0}$'s reduce to the Gelfand invariants [4] of the Lie algebra g, and so we call them the generalized Gelfand invariants of the quantum group $U_q(g)$.

4. Generalized Gelfand invariants for $U_q(so(5))$

The Cartan matrix for so(5) is

$$A=\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

We choose the bilinear form $(\cdot,\cdot)$ on H^* such that for the simple roots α_1 and α_2, $(\alpha_1,\alpha_1)=2(\alpha_2,\alpha_2)=2$. The Weyl group element of so(5) which maps all the positive roots into negative ones is $\sigma_0=\sigma_1\sigma_2\sigma_1\sigma_2$ where σ_1 and σ_2 are the elementary Weyl reflections with respect to the simple roots α_1 and α_2 respectively. Following the prescription given in [5], the $E(p)$, $F(p)$, $(p=1,2,3,4)$ for $U_q(so(5))$ can be obtained and we have

$$\begin{aligned}
&E(1)=E_1\\
&E(2)=-q^{-1}[E_1,E_2]_{q^{-1/2}}\\
&E(3)=q^{-1/2}[[E_1,E_2]_{q^{-1/2}},E_2]/(q^{1/2}+q^{-1/2})\\
&E(4)=E_2\\
&F(1)=F_1\\
&F(2)=[F_1,F_2]_{q^{-1/2}}\\
&F(3)=q^{-1/2}[[F_1,F_2]_{q^{-1/2}},F_2]/(q^{1/2}+q^{-1/2})\\
&F(4)=F_2
\end{aligned}\tag{11}$$

with

$$E_1=q^{h_1/2}e_1 \qquad F_1=q^{-h_1/2}e_2$$
$$E_2=q^{h_2/2}e_2 \qquad F_2=q^{-h_2/2}f_2.$$

In (11), we have used the q-brackets defined as follows

$$[A,B]_q=qAB-q^{-1}BA.$$

Inserting (11) in (2) we arrive at the following universal R-matrix of $U_q(so(5))$

$$\begin{aligned}
R=q^{(h_1+h_2)\otimes(h_1+h_2)+h_2\otimes h_2}&\sum_{\{n\}}\frac{(1-q^{-2})^{n_1+n_3}(1-q^{-1})^{n_2+n_4}}{(n_1)_q!(n_2)_{q^{1/2}}!(n_3)_q!(n_4)_{q^{1/2}}!}\\
&\times(E(1))^{n_1}(E(2))^{n_2}(E(3))^{n_3}(E(4))^{n_4}\\
&\otimes(F(1))^{n_1}(F(2))^{n_2}(F(3))^{n_3}(F(4))^{n_4}.
\end{aligned}\tag{12}$$

The smallest non-trivial representation of $U_q(so(5))$ is that obtained by deforming the spinor representation of so(5), which is four-dimensional with highest weight $\Lambda_0=\frac{1}{2}(\alpha_1+2\alpha_2)$. We denote this representation by π_{Λ_0} and take it as the reference representation. Observe that π_{Λ_0} is a minimal representation; a common feature of such representations is

$$[\pi_{\Lambda_0}(E(p))]^2=[\pi_{\Lambda_0}(F(p))]^2=0 \qquad \forall p.$$

It is straightforward to calculate $\pi_{\Lambda_0}(e_i)$, $\pi_{\Lambda_0}(f_i)$, $i=1,2$, and we have

$$\begin{aligned}
&\pi_{\Lambda_0}(e_1)=\begin{pmatrix}\mathbf{0} & \sigma_-\\ \mathbf{0} & \mathbf{0}\end{pmatrix} \qquad \pi_{\Lambda_0}(f_1)=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \sigma_+ & \mathbf{0}\end{pmatrix}\\
&\pi_{\Lambda_0}(e_2)=\begin{pmatrix}\sigma_+ & \mathbf{0}\\ \mathbf{0} & \sigma_+\end{pmatrix} \qquad \pi_{\Lambda_0}(f_2)=\begin{pmatrix}\sigma_- & \mathbf{0}\\ \mathbf{0} & \sigma_+\end{pmatrix}
\end{aligned}\tag{13}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and where $\mathbf{0}$ denotes a 2×2 block of zeros. Also

$$\begin{aligned}
&\pi_{\Lambda_0}(E(1)) = q^{1/2}\begin{pmatrix} \mathbf{0} & \sigma_- \\ \mathbf{0} & \mathbf{0} \end{pmatrix} && \pi_{\Lambda_0}(F(1)) = q^{1/2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \sigma_+ & \mathbf{0} \end{pmatrix} \\
&\pi_{\Lambda_0}(E(2)) = q^{1/4}\begin{pmatrix} \mathbf{0} & \begin{matrix} 1 & 0 \\ 0 & -q^{-1} \end{matrix} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} && \pi_{\Lambda_0}(F(2)) = q^{1/4}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \begin{matrix} 1 & 0 \\ 0 & -q \end{matrix} & \mathbf{0} \end{pmatrix} \\
&\pi_{\Lambda_0}(E(\Lambda)) = q^{1/2}\begin{pmatrix} \mathbf{0} & \sigma_+ \\ \mathbf{0} & \mathbf{0} \end{pmatrix} && \pi_{\Lambda_0}(F(3)) = q^{1/2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \sigma_- & \mathbf{0} \end{pmatrix} \\
&\pi_{\Lambda_0}(E(4)) = q^{1/4}\begin{pmatrix} \sigma_+ & \mathbf{0} \\ \mathbf{0} & \sigma_+ \end{pmatrix} && \pi_{\Lambda_0}(F(4)) = q^{1/4}\begin{pmatrix} \sigma_- & \mathbf{0} \\ \mathbf{0} & \sigma_- \end{pmatrix}.
\end{aligned} \tag{14}$$

Now substitute (14) in the formula (5). Lengthy but straightforward manipulations lead to

$$\begin{aligned}
C_L^{\Lambda_0} = {} & \frac{1}{(q-q^{-1})^2}\{q^2(q^{h_1+2h_2}-1)+q(q^{h_1}-1)+q^{-1}(q^{-h_1}-1)+q^{-2}(q^{-h_1-2h_2}-1)\} \\
& + qF(1)E(1) + q^{1/2}\frac{q^{h_2+1/2}+q^{-h_2-1/2}}{(q^{1/2}+q^{-1/2})^2}F(2)E(2) \\
& + F(3)E(3) + q^{-1/2}\frac{q^{h_1+h_2+3/2}+q^{-h_1-h_2-3/2}}{(q^{1/2}+q^{-1/2})^2}F(4)E(4) \\
& + \frac{q^{1/2}-q^{-1/2}}{q^{1/2}+q^{-1/2}}\{-q^{-h_2+1/2}F(1)F(4)E(2) + F(2)F(4)E(3) \\
& + q^{h_2+1/2}F(2)E(1)E(4) - qF(3)E(2)E(4)\} \\
& - q\left(\frac{q^{1/2}-q^{-1/2}}{q^{1/2}+q^{-1/2}}\right)^2 F(2)F(4)E(2)E(4).
\end{aligned} \tag{15}$$

In the limit $q\to 1$, the last two terms in (15) vanish and, because the Dynkin index of the spinor representation of so(5) is 1, the other terms produce $\frac{1}{2}C_L$, where C_L is the quadratic Casimir operator for so(5). This is in agreement with the general result stated earlier.

Using (6) we can work out the operator A which commutes with $\Delta(\mathrm{U}_q(\mathrm{so}(5)))$, and in turn we can construct the higher order central elements $I_m^{\Lambda_0}$ of $\mathrm{U}_q(\mathrm{so}(5))$ by utilizing equation (8). However, it turns out that both $I_2^{\Lambda_0}$ and $I_3^{\Lambda_0}$ are linear combinations of $C_L^{\Lambda_0}$ given in (15) and constant terms; a new central element arises from $I_4^{\Lambda_0}$, which, containing terms like $F(2)F(4)E(2)E(4)F(3)E(2)E(4)F(1)F(4)E(2)$ etc, is very messy, thus we do not spell it out explicitly here.

5. Concluding remarks

An important question that remains to be answered is whether the $I_m^{\Lambda_0}$'s generate the centre of the quantum group $U_q(g)$. We expect the answer to be affirmative, and hope to present a detailed study of this question elsewhere.

Finally we want to mention that a different set of central elements can be obtained using the general method of [1, 10], namely

$$C_m^{\Lambda_0} = \mathrm{tr}_{\pi_{\Lambda_0}}[(\mathrm{id} \otimes \pi_{\Lambda_0})(1 \otimes q^{2h_\rho})\Gamma^m] \qquad m \in \mathbb{Z}$$

where $\Gamma = R^T R$. The eigenvalues of the $C_m^{\Lambda_0}$'s can also be easily computed, and it is very likely that the $C_m^{\Lambda_0}$, $m \geq 1$, also generate the centre of $U_q(g)$.

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